Quantitative and Qualitative Analysis of Autonomous ODE's
7/13/18

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1 Overview

This lecture covers most of sections 1.3 and 1.5 in Teschl's book. If there is time, 1.4 will be covered as well. The first part of the lecture (section 1.3) deals with the existence and uniqueness of autonomous ODE's, given certain restrictions on the function $f(x) = \dot{x}$. The second part introduces the qualitative approach to ODE analysis, i.e. understanding the long-term behavior of solutions of ODE's, without finding explicit solutions. If there is time, we will go over some formulas for specific ODE's (section 1.4).

2 Section 1.3

In the last lecture, Nikhil established that any ODE can be represented by the form

$$\dot{x} = f(x,t),\ x(0) = x_0$$

It will be shown that one way of getting existence and uniqueness of solutions is by having the condition that f is independent of t, i.e $\frac{\partial f}{\partial t} = 0$ which is the definition of an *autonomous* ODE. Further, we restrict our choice of f such that $f \in C^0(\mathbb{R}, \mathbb{R})$.

We are therefore looking for solutions to

$$\dot{x} = f(x),\ x(0) = x_0 \tag{2.0.1}$$

Now, suppose that $f(x_0) \neq 0$. Then, due to continuity, there exists a maximal open interval $I = (x_1, x_2) \ni x_0$ such that $f(x) > 0$ for $x \in I$. Note that $x_1, x_2 \in [-\infty, \infty]$. On I, we can divide $f(x)$ from both sides of (2.0.1) and integrate with respect to t to get

$$\int_0^t \frac{\dot{x}(s)ds}{f(x(s))} = t$$

A substitution of variables gives us

$$\int_{x(0)}^{x(t)} \frac{du}{f(u)} = t$$

Defining $F(x(t)) \equiv \int_{x(0)}^{x(t)} \frac{du}{f(u)}$, we can see that if $\varphi(t)$ is a solution to (2.0.1), then $F(\varphi(t)) = t$ as well. This is very convenient, because all we need to do now is invert F to get an explicit solution for φ.

Since $f(x) > 0$ on I, we have that if $x(t_1) > x(t_2)$ in I, then $\int_{x(0)}^{x(t_1)} \frac{du}{f(u)} > \int_{x(0)}^{x(t_2)} \frac{du}{f(u)}$, implying that F is strictly increasing. This now gives us that F has a unique and well defined C^1 inverse function $F^{-1} : \mathbb{R} \to \mathbb{R}$. Defining $\varphi(t) \equiv F^{-1}(t)$, we now have existence and uniqueness of a solution.

A natural question arises: What is the largest interval on which φ is defined? The answer lies in the range of F. Define the following two numbers:

$$T_+ = \lim_{x \to x_2} F(x),\ T_- = \lim_{x \to x_1} F(x)$$

Note that $T_+ \in (0, \infty]$, since $F(x_0) = 0$ and $F(x) > F(x_0)$ for $x > x_0$, and similarly, $T_- \in [-\infty, 0)$. Since F is increasing, the range of F is (T_-, T_+), and so $\varphi \in C^1((T_-, T_+))$.

Let examine whether or not we can extend solutions beyond (T_-, T_+). If $T_+ < \infty$ while $x_2 = \infty$, then we cannot not extend, because φ approaches infinity at T_+. The only other possible case is when $f(x_2) = 0$, in which case we could just set $\varphi(t) = x_2$ for $t > T_+$. Similar reasoning holds for when T_- is finite.

Note that if $f(x_0) = 0$ then the solution may still exist, but it may not be unique.

Example 2.1. Consider $f(x) = \sqrt{x}$, where $x(0) = x_0$.
If we assume at first that $x_0 > 0$, then $f(x_0) > 0$. So, we can compute

$$F(x) = \int_{x_0}^{x(t)} \frac{dx}{\sqrt{x}} = 2(\sqrt{x} - \sqrt{x_0})$$

Therefore if $F(\varphi(t)) = t$, then

$$\begin{aligned} 2(\sqrt{\varphi(t)} - \sqrt{x_0}) &= t \\ \varphi(t) &= \Big(\frac{t + 2\sqrt{x_0}}{2}\Big)^2 \end{aligned}$$

This is defined on the interval $(-2\sqrt{x_0}, \infty)$. Now, if $x_0 = 0$, then $\overline{\varphi}(t) = \begin{cases} -\left(\frac{t}{2}\right)^2 & \text{if } t < t_1 \\ 0 & \text{if } t \in (t_1, t_2) \\ \left(\frac{t}{2}\right)^2 & \text{if } t > t_2 \end{cases}$
is a solution, for any $t_1 < 0$ and $t_2 > 0$.

3 Section 1.5

Previously, we found explicit solutions to ODE's, but often times, $F(x)$ is very difficult to compute. In this section, we see that an explicit solution is not need to answer questions regarding the long-time behavior of the system. Let us begin with an example

Example 3.1. The logistic growth model is given as $\dot{x} = -x^2 + x - h = f(x)$, where $x : \mathbb{R} \to \mathbb{R}$. I'm not exactly sure how to properly draw a graph in Latex, but it's in the textbook. The basic idea is that we draw the graph of the function on a plane, and observe where the function is positive, negative and zero.

Key takeaways from this example in textbook:
At positive points of f, $x(t)$ increases, and at negative points $x(t)$ decreases, and if $x(t)$ reaches a point r at time t_r such that $f(r) = 0$, then $x(t) = r$ for all $t > t_r$. Further, if WLOG $f(x_0) > 0$, (where x_0 is the initial condition) and there is no $r > x_0$ such that $f(r) = 0$, then as $t \to \infty$, $x(t) = \infty$. This makes sense, because the derivative of any solution $x(t)$ given the initial condition x_0 would be positive for all time.

What if our differential equation is not-autonomous? This type of qualitative analysis is still possible.

Example 3.2. Consider the ODE: $\dot{x} = x^2 - t^2 = f(x,t)$. We know this function is C^1, so it's locally Lipschitz and a unique solution exists. However, we will not try to explicitly solve it and see what we can deduce just be looking at the level curves of the function f. Again, I don't quite know how to draw graphs on Latex, but they are in the textbook.
This time, we consider, only for $t \geq 0$, the regions of the (x,t) plane for which $f(x,t)$ is positive and negative. These regions are split by the graphs $x = t$ and $x = -t$. By graphical analysis, it becomes clear that there are some regions for which if a solution starts in that region, it cannot leave that region due to the sign of f. Further, there are regions for which all solutions starting in that region must leave that region for another. As a result of these sort of findings, one can determine the long-time behavior of solutions depending on initial conditions (t_0, x_0).

There are more questions we can answer with the qualitative approach:

- Do solutions converge to their limits in finite or infinite time?
- Which solutions do/don't converge to the level curve $f(t,x) = 0$?

In order to answer these questions, we need to introduce the concept of a *super* and *sub* solution.

Definition 3.3. A super solution is a differentiable function $x_+(t)$ such that $\dot{x}_+(t) > f(t,x)$ in a certain interval $[t_0, T)$

Definition 3.4. Similarly, a sub solution is a differentiable function $x_-(t)$ such that $\dot{x}_+(t) < f(t,x)$ in a certain interval $[t_0, T)$

Example 3.5. The functions $x_+(t) = t$ and $x_-(t) = -t$ are respectively super and sub solutions of the ODE in ex. 3.2. The whole point of these definitions are to capture the role that these functions played in the previous exercise.

The importance of the super and sub solutions can be understood by the following lemma:

Lemma 3.6. *Let $x_+(t)$ and $x_-(t)$ be super and sub solutions of the differential equation $\dot{x} = f(x,t)$ on $[t_0, T)$. Then for every solution $x(t)$, we have that:*

$$\textit{If } x(0) < x_+(0), \textit{ then } x(t) < x_+(t) \textit{ for } t \in (t_0, T)$$
$$\textit{If } x(0) > x_-(0), \textit{ then } x(t) > x_-(t) \textit{ for } t \in (t_0, T)$$

Proof. Define $\Delta(t) = x_+(t) - x(t)$. Since by assumption, $x_+(t_0) \geq x(t_0)$, we see that $\Delta(t_0) \geq 0$. Further,

$$\dot{\Delta}(t) = \dot{x}_+(t) - \dot{x}(t) > f(t,x) - f(t,x) = 0$$

Because in the interval (t_0, T), we have that $\dot{x}_+(t) > f(t,x)$, while $\dot{x}(t) = f(t,x)$. This proves the first part of the lemma. The second part follows similarly. □

Now that we can see the importance of super and sub solutions, the name of the game seems to be: How can I find increasingly restrictive super and sub solutions to better understand the convergence of solutions? This can be done by finding other level curves of $f(t,x)$, called isoclines, and seeing whether or not these isoclines are super or sub solutions.

Definition 3.7. An isocline of f is the set of points where $f(t,x) = c$ for a particular $c \in \mathbb{R}$.

For basically the rest of the section, the textbook just deals with example 3.2. (i.e. $\dot{x} = x^2 - t^2$). Consider the isocline $f(t,x) = -2$, or $x^2 - t^2 = -2$. Solving for x, we get $x = \pm\sqrt{t^2 - 2}$, but just consider for now

$$x = -\sqrt{t^2 - 2}$$

. We can check that is in fact a super solution $y_+(t)$, because

$$\dot{y}_+(t) = \frac{-t}{\sqrt{t^2 - 2}} > -2 = f(t, x(t))$$

If $f > 2\sqrt{\frac{2}{3}}$. Now, it is straightforward to check two things:

- $y_+(t)$ converges to $x_-(t) = -t$
- $y_+(t) > x(t) = -t$ for a certain interval

Therefore, if a solution $\varphi(t)$ is such that $x_-(t) \leq \varphi(t_0) \leq y_+(t)$, then $\varphi(t)$ must converge to $x_-(t)$ as well! A natural follow-up question is: Does every solution between $x_-(t) = -t$ and $x_+(t) = t$ end up in-between $x_-(t)$ and $y_+(t)$? The answer is yes. Consider the function $-y_+(t)$. Since $x_-(t) < -y_+(t) < x_+(t)$, $f(t,x)$ is decreasing (on either) of $y_+(t)$. Therefore every solution starting between $x_-(t)$ and $x_+(t)$ will eventually be below $-y_+(t)$. Further, for all solutions $x(t)$ in-between $y_+(t)$ and $-y_+(t)$, we can calculate that $\dot{x}(t) < -2$, which means that the slope of $\dot{x}(t)$ is steeper than the line $x_-(t)$, meaning that eventually, $x(t)$ will enter the region between $y_+(t)$ and $x_-(t)$, and so it will eventually converge to $x_-(t)$.

We have just shown that if the solution $\varphi(t)$ is such that $x_-(t) < \varphi(t) < x_+(t)$, then eventually $\varphi(t)$ will converge to $x_-(t)$.

Now, let us think about the region where $x(t) > x_+(t) = t$. Consider the isocline $f(t, x) = 2$, and in particular, $y_(t) = \sqrt{2+t^2}$ which turns out to be a sub solution. Since $y_-(t) > x_+(t)$, we look at solutions that stay within the region between $y_-(t)$ and $x_+(t)$ for finite time $t \in (0, T)$. Look at the solutions with initial conditions $(T, x_+(T))$ and $(T, y_+(T))$ at time $t = 0$. These two solutions diverge from each other, as one crosses x_+, and the other crosses y_-. By looking at the graph, it becomes apparent that the solutions that are in this region for at least time T are between the solution curves with the initial conditions $(T, x_+(T))$ and $(T, y_+(T))$. Interestingly, above x_+, $\partial f(t, x) \partial t > 0$, meaning that the distance between solutions increases with time. However, we can verify that y_- converges to x_+. These two pieces of information tell us that for any bunch of solutions in the region between y_- and x_+, at most one can remain in this region, i.e. converge to x_+.

We finally show that if a solution $x(t) > y_(t)$, then $x(t) \to \infty$ in finite time. For every such $x(t)$, we know that $\dot{x}(t) > 2$. Therefore, $x(t) > 2(t - t_0) + x_0$. This in turn means that there exists $\epsilon > 0$ such that

$$x(t) > \frac{2}{\sqrt{1-\epsilon}}$$

Therefore, $\dot{x}(t) > x(t)^2 - (1-\epsilon)t^2 = \epsilon x^2$

Now, we already know that the function $\dot{x}(t) = \epsilon x^2$ diverges to infinity in finite time, and so we are done, because our solution lies above the solution for which $\dot{x}(t) = \epsilon x^2$.Note that this approach only works if $f : \mathbb{R}^2 \to \mathbb{R}$.