WALL'S OBSTRUCTION THEOREM

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Abstract. This paper discusses Wall's Finiteness Obstruction Theorem, which addresses when a CW complex that is dominated by a finite complex has the homotopy type of a finite complex. We first introduce the theorem using a more geometric setting to provide visual intuition, discussing necessary and sufficient conditions for finite domination before stating and proving the theorem. Then, we state the analogous theorem for chain complexes, which has allowed for the utilization of obstruction theory in other areas of mathematics. We conclude by proving a well-known analogue of Wall's theorem for G-CW complexes, or complexes with a finite group action.

Contents

1. Introduction

Finiteness obstruction plays an important role in dealing with problems related to classifying manifolds. In high-dimensional topology, when proving two manifolds are homeomorphic, one often decomposes them into smaller, more manageable homeomorphic manifolds. Finiteness obstructions must usually be overcome in this process. We provide further motivation for the topic of the paper by giving examples of problems where finiteness obstruction appears in their solutions (or partial solutions):

(1) (The Space-Form Problem): For any $n \in \mathbb{N}$, can we classify every smooth manifold with S^n as a universal cover?
(2) (The Triangulation Problem): Is every compact topological manifold without boundary homeomorphic to a finite polyhedron?
(3) When is a given smooth open manifold the interior of a smooth compact manifold with boundary?

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Section 2 provides some background in algebraic topology for the later sections (knowledge of undergraduate-level algebraic topology is assumed). Section 3 gives us an overview of Wall's results using geometric language, Section 4 rephrases parts of Section 3 in terms of homological algebra, which will allow for the application of obstruction theory in any homology theory.

2. Algebraic Topology: Prerequisites

For the entirety of the paper, we assume our spaces have the homotopy types of connected CW-complexes. If X is such a space, then $\tilde{X}$ denotes its universal cover.

2.1. **The Homotopy Groups of a Map.**

Definition 2.1. Consider a map $\varphi : X \to Y$. We can define a new space, called the *mapping cylinder*, to be the quotient space

$$M\varphi := \Big((X \times I) \coprod Y\Big)/ \sim$$

The equivalence relation is given by $(0, x) \sim f(x)$ for all $x \in X$. Further, $\coprod$ denotes the disjoint union, and I is the unit interval.

Loosely speaking, the mapping cylinder is the space obtained by "gluing" Y to one end of $X \times I$ by the map φ. Note that M_f can be deformation retracted to Y by collapsing one end of $X \times I$ into the other, which is identified by φ as a subspace of Y. Using this construction, we can define the homotopy groups of φ.

Definition 2.2. Given a map $\varphi : X \to Y$, we define $\varphi_n(\varphi)$ to be the following relative homotopy groups:

$$\pi_n(\varphi) := \pi_n(M\varphi, X \times \{1\})$$

Similarly, we can define

$$H_n(\varphi) := H_n(M\varphi, X \times \{1\})$$

By our knowledge of long exact sequences of homotopy groups, we can observe that we have the following exact sequence:

$$\cdots \to \pi_n(X) \to \pi_n(M\varphi) \to \pi_n(\varphi) \to \pi_{n-1}(X) \to \dots$$

We now conclude this subsection with the following definition, which is used extensively in the paper:

Definition 2.3. A map $\varphi : X \to Y$ is *n-connected* if $\pi_i(\varphi) = 0$ for $0 \leq i \leq n$.

Observe that φ must be n-connected if $\varphi_* : \pi(X) \to \pi(Y)$ is an isomorphism for $i < n$ and a surjection for $i = n$.

2.2. **The Homotopy Fibration Associated to a Map.**

The goal of constructing a homotopy fibration is to associate a fibration to every continuous map.

Definition 2.4. Given any map $\varphi : X \to Y$, we define the mapping path space $N\varphi = \{(x, \gamma_x) | x \in X, \gamma_x : I \to Y \text{ such that } \gamma(0) = \varphi(x)\}$. Further, $N\varphi$ is given the subspace topology of $X \times Y^I$ where Y^I has the compact open topology.

Observe that $N\varphi$ deformation retracts onto X, which can be identified as $\{(x, c_x)\}$, where c_x is the constant path at x. To see this retraction, shrink each curve γ_x to c_x. This allows us to construct the following map:

Proposition 2.5. *For every map $\varphi : X \to Y$, we can associate the fibration $p : N\varphi \to Y$ such that $p(x, \gamma_x) = \gamma(1)$. Further, if we assign a base point $y_0 \in Y$, we can define the homotopy fiber $F_\varphi := \{(x, \gamma_x) \in E_f \mid \gamma(1) = y_0\}$.*

We will use the following proposition in the next section, and we will state it without proof:

Proposition 2.6. *Given a map φ and the associated fibration $p : N\varphi \to Y$, we get the isomorphism $\pi_i(p) \cong \pi_{i-1}(F_f)$.*

2.3. **The Action of $\pi_1(X)$ on $\pi_n(X)$ and $C_n(\tilde{X})$.** Understanding the $\mathbb{Z}[\pi_1]$ module structure on π_n is imperative to the geometric formulation of Wall's finiteness theorem. In order to understand how $\pi_1(X, *)$ acts on $\pi_n(X, *)$ of a space, we first need to see how any map $\alpha : I \to X$ such that $\alpha(0)$ can be used to construct a new map

$$\pi_n(X, \alpha(0)) \to \pi_n(X, \alpha(1))$$

Proposition 2.7. *Consider such a map α. Then, we can define a map $\pi_n(X, \alpha(0)) \to \pi_n(X, \alpha(1))$ in the following way:*
Take a an element $f : (I^n, \partial I^n) \to (X, \alpha(0))$, and associate it to the map $\gamma f : (I^n, \partial I^n) \to (X, \alpha(1))$ such that we shrink the domain of f to a concentric cube J^n strictly inside I^n, and perform γ on each radial segment between J^n and I^n as can be seen in this picture:

It can be observed that this map is only dependent on the homotopy class of α. Further, in a path-connected space, we can use this proposition to observe that for any $x_0, x_1 \in X$, $\pi_n(X, x_0) \cong \pi_n(X, x_1)$. We are now in a position to describe the action of $\pi_1(X, *)$ on $\pi_n(X, *)$, simply by restricting our attention to curves α with the same starting and ending point.

Proposition 2.8. *There is an action of $\pi_1(X, *)$ on $\pi_n(X, *)$ is given by:*

$$[\alpha : (I, \partial I) \to (X, *)][f : (I^n, \partial I^n) \to (X, *)] = [\alpha f : (I^n, \partial I^n) \to (X, *)]$$

This action is well-defined up to homotopy class. Note, of course, that we cannot call $\pi_n(X)$ a $\pi_1(X)$ module, because $\pi_1(X)$ is just a group. However, we can fix this using the concept of an integral group ring:

Definition 2.9. Given a ring R and a group G, the integral group ring, denoted $R[G]$, is a ring where the elements are those of the free R module with generating set G. Addition is defined by the free-module structure, and multiplication is defined by using the group law on the basis elements (i.e. $r_1 g_1 \times r_2 g_2 = r_1 r_2 (g_1 + g_2)$) and extending linearly.

Now, for $n \geq 2$, $\pi_n(X)$ is abelian, so we can now turn $\pi_n(X)$ into the module we want.

Proposition 2.10. *For $n \geq 2$, by the group action of $\pi_1(X)$ on $\pi_n(X)$, it follows that $\pi_n(X)$ is a $\mathbb{Z}[\pi_1(X)]$ module.*

Further, we know that $\pi_1(X, *)$ acts on the covering space $\tilde{X}$, and therefore π_1 will also act on the singular n-chains of $\tilde{X}$. So, we observe that this action turns $C_n(\tilde{X})$ into a $\mathbb{Z}[\pi_1(X)]$ module as well. Observe that, while these two actions seem different, we will see in the next section that they are closely connected by the Hurewicz homomorphism.

2.4. **Defining $\tilde{K}_0(\mathbb{Z}[\pi_1])$.** Consider a ring R, and denote S_R to be the set of isomorphism classes of projective R modules. Now, let $\mathcal{F}(S_R)$ be the free-abelian group generated by S. We can define an equivalence relation $\sim$ on $\mathcal{F}(S_R)$, where $[X \oplus Y] - [X] - [Y] \sim [0]$.

Definition 2.11. We define $K_0(R) := \mathcal{F}(S_R)/\sim$. Further, there is a canonical injective homomorphism $i : \mathbb{Z} \to R$, which extends in a natural way to $i^\# : K_0(\mathbb{Z}) \to K_0(R)$ such that $i^\#(1) = R$ (note that $K_0(\mathbb{Z}) \cong \mathbb{Z}$). We can define the *reduced projective class group* to be

$$\tilde{K}_0(\mathbb{Z}[\pi_1(X)]) = coker(i^\#)$$

The following is an exercise in definition chasing:

Exercise 2.12. Two R modules P and Q are in the same $\tilde{K}_0$ class iff there exist finitely generated free modules F, G such that $P \oplus F \cong Q \oplus G$.

We have now introduced all of the machinery necessary to understand the statements and the proofs in the next section.

3. A Geometric Understanding of Obstruction

Before arriving at the major theorem, it is necessary for understanding that we deal with the following intermediate questions:
When does a CW-complex have the homotopy type of:

(1) a complex with a finite n-skeleton?
(2) a complex with finite dimension?

Answering these two questions will further tell us when a CW complex is dominated (or the retract) of a finite complex (in both dimension and skeleta).

3.1. **A Finite n-skeleton?** We define the following properties for a space X, which we label as Fn:

F1: $\pi_1(X)$ is finitely generated.
F2: $\pi_1(X)$ is finitely presented, and for any finite 2-dimensional CW complex K^2 and a 1- connected map $\varphi : K^2 \to X$, $\pi_2(\varphi)$ is a finitely generated $\mathbb{Z}[\pi_1]$ module.
Fn: $F(n-1)$ holds, and for any $(n-1)$-connected map $\varphi : K^{n-1} \to X$, where K^{n-1} is a finite n- dimensional complex, $\pi_n(\varphi)$ is a finitely generated $\mathbb{Z}[\pi_1]$ module. The major theorem from this subsection is as follows:

Theorem 3.1. *X is homotopy equivalent to a CW complex with a finite n-skeleton iff X satisfies Fn.*

Proof. Suppose X is a complex with a finite one skeleton. Then, we can compute the edge path group (which is the same as the fundamental group) to see that π must be finitely generated. Further if X is a complex with a finite two-skeleton, we can still compute the edge path group and observe that π_1 is finitely presented. Suppose $n \geq 3$ and $\varphi : K^{n-1} \to X$ is an $n-1$ connected map and K^{n-1} is a finite complex. We can modify φ such that $im(\varphi) \subset X^{n-1}$, giving us the long exact sequence:

$$\cdots \to \pi_n(X, X^n) \to \pi_{n-1}(X^n, K^{n-1}) \to \pi_{n-1}(X, K^{n-1}) \to \dots$$

Noting that $\pi_{n-1}(\varphi)$ can be identified with $\pi_{n-1}(X, K^{n-1})$, we see that $\pi_{n-1}(X, K^{n-1}) = \cdots = \pi_1(X^n, K^{n-1}) = 0$. Further, $\pi_k(X, X^n) = 0$ for $k < n$, which implies that $\pi_k(X^n, K^{n-1}) = 0$ for $k < n$. So, the induced map from K^{n-1} to X^n is $n-1$ connected as well. We can now deformation retract the skeleton X^i into K^i (for $i < n$) via the mapping cylinder of φ, which is $n-1$ connected. Then, by the Hurewicz isomorphism:

$$\pi_n(\varphi) = \pi_n(X, K) \cong \pi_n(\tilde{X}, \tilde{K}) \cong H_n(\tilde{X}, \tilde{K})$$

Observe that $H_n(\tilde{X}, \tilde{K})$ is a quotient of $H_n(\tilde{X}^n, \tilde{K}) \cong H_n(\tilde{Y}^{n+1}, \tilde{K})$, which is in turn a quotient of $H_n(\tilde{Y}^n, \tilde{K}) \cong C_n(\tilde{Y})$. Further, $C_n(\tilde{Y})$ is a finitely generated $\mathbb{Z}[\pi_1]$ module due to the fact that Y is finite. Therefore, $\pi_n(\varphi)$ must also be finitely generated, proving the forward direction.

Now, for the reverse, suppose we have an $n-1$ connected map $\varphi : K^n \to X$. If $n \geq 3$, we know that $\pi_1(\varphi) = 0$ so we can have X adopt the one-skeleton of K^n, which turns $\pi_n(\varphi)$ into a $\mathbb{Z}[\pi_1]$ module. Since it is finitely generated, we choose the generators $\alpha_1, \ldots, \alpha_n$. Observe that $\partial\alpha_i \in \pi_{n-1}(K)$, and so we attach n cells along each boundary, and one can check that this extends φ to an n-connected map. This allows X to be homotopy equivalent to a CW complex with a finite n skeleton (where n depends on the number of generators of $\pi_n(\varphi)$). □

On its own, this theorem is not very insightful, because one does not know whether such $(n-1)$ connected maps φ can even be constructed. However, we hinted in the proof that we can construct such a map φ and a CW complex K in the following way:

Construction 3.2.
1. Building a 1-connected map: Looking at number of generators for the fundamental group of X (suppose it is n), then we can let K^1 be a bouquet of n circles. We define φ to map each circle to each generator, which gives us a surjection on π_1. In the previous section, we observed that this makes φ 1-connected.
2. If $\varphi : K \to X$ is $(n-1)$ connected, we view $\pi_n(\varphi)$ as a $\mathbb{Z}[\pi_1]$ module. Choosing the generators $\{\alpha_i\}_{i \in I}$, we observe that $\partial\alpha_i \in \pi_{n-1}(K)$, and so we can attach n cells along each boundary in K, both constructing a skeleton K^n and extending φ to an n-connected map.

3.2. **A Finite Dimensional Complex?** Considering the map $\varphi : K \to X$ which we built inductively in the previous section, the question arises: In what cases can we stop attaching i cells for $i > n$ and obtain a homotopy equivalence? The necessary and sufficient conditions are denoted by Dn. We deal with $D1$ separately:

Definition 3.3 (Property D1)**.** $H_i(\tilde{X}) = 0$ for $i > 1$, and for any representation of $\pi_1(X) \to End(G)$,

$$Ext^1_{\pi_1(X)}(\pi_1(X), G) = 0$$

Where $Ext^1_G(A, B) := \{P : 0 \to A \to P \to B \to 0$ is an exact sequence of preserving the G group action $\}$.

For $n \geq 2$, we have the following properties:

Definition 3.4 (Property Dn, n $\geq$ 2)**.** $H_i(\tilde{X}) = 0$ for $i > n$, and for any *abelian* group G with an action of $\pi_1(X)$:

$$H^{n+1}\Big(Hom_{\pi_1(X)}(C_*(\tilde{X}), G)\Big) = 0$$

$C_*(\tilde{X})$ is the singular complex of $\tilde{X}$, and $Hom_{\pi_1(X)}\Big(C_*(\tilde{X}), G)\Big)$ is the cochain complex where an n-cochain is a group action preserving map from $C_n(\tilde{X})$ to G. As a reminder, the boundary maps are given by considering the normal boundary map $\partial_n : C_n(\tilde{X}) \to C_{n-1}(\tilde{X})$ and defining $d_n : Hom_{\pi_1(X)}\Big(C_{n-1}(\tilde{X}), G)\Big) \to Hom_{\pi_1(X)}\Big(C_n(\tilde{X}), G)\Big)$ by $d_n(\varphi)(x) = \varphi(\partial_n x)$.

These properties give us the theorem:

Theorem 3.5. *X is homotopy equivalent to a CW complex of dimension n iff X satisfies Dn.*

For all $n \in \mathbb{N}$, the forward direction is clear. We first look at the reverse direction for $n = 1$ and prove this separately.

Lemma 3.6. *If X satisfies $D1$, then it is homotopic to a bouquet of circles.*

Proof. Since $H_1(\tilde{X}) = 0$ as well, it follows by Whitehead's theorem that $\tilde{X}$ is contractible, and so X is a $K(\pi_1(X), 1)$ space as it has no higher homotopy groups. Now, $Ext^1_{\pi_1(X)}(\pi_1(X), G) = 0$ implies that there are no such action preserving extensions of G by $\pi_1(X)$ except for the direct sum. Therefore, given a group homomorphism from a free group $F \to \pi_1(X)$ with kernel G, we observe that this map must split. Necessarily, $\pi_1(X)$ must be isomorphic to a subgroup of a free group, and is therefore free. This implies that X has the homotopy type of a bouquet of circles. □

Now, we can begin the proof for $n \geq 2$. We start with a lemma:

Lemma 3.7. *Consider $\varphi : K \to X$ as constructed in the last subsection (K is $n-1$ dimensional), where X is a space satisfying Dn. Then, $\pi_n(\varphi)$ is a* projective *$\mathbb{Z}[\pi_1]$ module.*

Proof. We note that by the aforementioned construction, we can simply assume that $X^i = K^i$ for $i < n$. Recall that

$$\pi_n(\varphi) = \pi_n(X, K) \cong H_n(\tilde{X}, \tilde{K}) \cong ker(\partial_n)/im(\partial_{n+1}) = C_n(\tilde{X})/B_n(\tilde{X}) := C_n/B_n$$

C_n denotes the set of singular n chains of $\tilde{X}$ and B_n denotes the boundary, i.e. $im(\partial_{n+1})$. Observe that every boundary of C_n is in $\tilde{K}$ (since K is the lower skeleton of X). Noting that C_n is free, we need to show that the short exact sequence $B_n \to C_n \to \pi_n(\varphi)$ splits to deduce that $\pi_n(\varphi)$ is projective. If we consider the map

$$\partial_{n+1} : C_{n+1} \to B_n$$

it follows that the map is a cochain of $Hom_{\pi_1(X)}(C_{n+1}(\tilde{X}), G)$ with coboundary $\partial_{n+1} \circ partial_{n+2} = 0$, since $\partial^2 = 0$. By assumption $H^{n+1}\Big(Hom_{\pi_1(X)}(C_*(\tilde{X}), G)\Big) = 0$, so it must also be a cocycle, meaning that there exists $s : B_n \to C_n$ such that $\partial_{n+1} = s \circ i \circ \partial_{n+1}$, where $i : B_n \to C_n$ is the left map in the short exact sequence. Since ∂_{n+1} as defined as onto (we purposely restricted the map to its image, i.e. B_n) we see that $s \circ i = id$, implying that the sequence splits. Therefore, we have shown that $\pi_n(\varphi)$ is projective. □

Suppose is $\pi_n(\varphi)$ is just a free module. Then, we get the following result:

Lemma 3.8. *Suppose $\varphi : K \to X$ is an $(n-1)$ connected map, where K is an $n-1$ complex and $\pi_n(\varphi)$ is free. If we add n cells to K to construct n-dimensional complex Y making φ n-connected, it follows that Y is homotopy equivalent to X.*

Proof. Just like in the previous proof, we can take Y as a subcomplex of X. Immediately, $H_i(\tilde{Y}, \tilde{K}) = 0$ if $i < n$ since K is all of the i skeletons of Y for $i < n$, and $H_i(\tilde{Y}, \tilde{K}) = 0$ for $i > n$ since Y is n dimensional.

Further observe that $H_n(\tilde{Y}, \tilde{K}) = ker(\partial_{n+1}) = C_n(\tilde{Y})$. Similarly, $H_i(\tilde{X}, \tilde{K}) = 0$ for $i \neq n$ (this holds for $i > n$ because X satisfies Dn). Since we constructed Y to extend φ to an n connected map, it follows that

$$H_n(\tilde{Y}, \tilde{K}) \cong H_n(\tilde{X}, \tilde{K}) \cong \pi_n(\varphi) = 0$$

Therefore, by the long exact sequence

$$\cdots \to H_n(\tilde{X}, \tilde{Y}) \to H_{n-1}(\tilde{Y}, \tilde{K}) \to H_{n-1}(\tilde{X}, \tilde{K}) \ldots$$

It follows that $H_i(\tilde{X}, \tilde{Y}) = 0$ for all $i \in \mathbb{N}$, which implies that the map φ induces an isomorphism on all homotopy groups, and by Whitehead's theorem we get a homotopy equivalence. $\square$

Now, we can prove the major theorem of this subsection.

Proof. If X satisfies and Dn and Fk for $k \leq n-2$, then using the usual construction, we get an $n-1$ connected map $\varphi : K \to X$ such that K is an $n-1$ dimensional complex with a finite k-skeleton for $k \leq n-2$. By the previous lemma, $\pi_n(\varphi)$ is a projective $\mathbb{Z}[\pi_1]$ module, implying that there exists a module M such that $\pi_n(\varphi) \oplus M \cong F$, where F is a free module. Now, we can construct the infinite-dimensional free module:

$$F' := [\pi_n(\varphi) \oplus M] \oplus [\pi_n(\varphi) \oplus M] \oplus \cdots = F \oplus F \oplus \cdots = \pi_n(\varphi) \oplus F'$$

For each generator of F' we attach to K^{n-1} a copy of an $n-1$ sphere to get a new complex Y, and extend φ to a map φ' by sending each of these spheres to the base point at which $\pi_1(X)$ is defined. Now, observe that we can turn the exact sequence

$$\cdots \to \pi_n(\varphi) \to \pi_n(\varphi') \to \pi_n(Y, K) = F \to \ldots$$

into:

$$\cdots \to H_n(\tilde{X}, \tilde{K}) \to H_n(\tilde{X}, \tilde{Y}) \to H_n(\tilde{Y}, \tilde{K}) \to \ldots$$

Since Y dominates K, the sequence splits, giving us that $\pi_n(\varphi') \cong \pi_n(\varphi) \oplus F' = M \oplus F' = F'$. So $\pi_n(\varphi')$ is free. We can follow the preceding lemma to construct a finite n- dimensional complex that gives us homotopy equivalence. $\square$

3.3. **Domination and Obstruction.** Now, we combine the previous two subsections to answer the question: When is a CW-complex finitely dominated (i.e. both finite skeleta and finite dimension)?

Theorem 3.9. *A connected CW-complex X is finitely dominated by an n complex iff X satisfies Dn and Fn.*

Proof. The reverse direction is clear. For the forward direction, we construct an n-connected map $\varphi : K \to X$, where K is a finite n-complex. Now, construct the associated fibration $p : N\varphi \to X$ and recall that if F_f is the fiber then $\pi_i(p) \cong \pi_{i-1}(F_f)$. Equivalent to finding a homotopy right inverse to φ, we can find a section of the fibration. Notice that $H^i(X, \pi_i(p)) = 0$ since $\pi_i(p) = \pi_i(\varphi) = 0$ for $i \leq n$. Further, $H^i(X, \pi_i(p)) = 0$ for $i > n$ since X satisfies Dn (and therefore Di). As a result, there will be no obstructions to finding a section, and so a section will exist, giving us a right homotopy inverse. Therefore, K dominates X. Further if X satisfies Fn, we can make K satisfy Fn, and therefore make sure it has finite skeleta in all dimensions. Therefore, X is finitely dominated. $\square$

Now, observe that if X is finitely n-dominated, the previous two subsections have informed us that if we look at $\pi_n(\varphi)$, where $\varphi : K \to X$ is n-connected, then it is not only finitely generated, but also a projective

$\mathbb{Z}[\pi_1]$ module. Therefore, $\pi_n(\varphi)$ must define an element of $\tilde{K}_0(\mathbb{Z}[\pi])$. A natural question arises: do different $n-1$ connected maps define different elements $\tilde{K}_0(\mathbb{Z}[\pi])$? The answer is no; a proof can be found in [1], and we have left it out of the paper as it does not provide further intuition. Wall's finiteness obstruction can now be proven, concluding this section.

Theorem 3.10. *If X is a complex finitely dominated by an n complex K with an $(n-1)$-connected map φ, then X has the homotopy type of a finite complex iff*

$$[\pi_n(\varphi)] \in \tilde{K}_0(\mathbb{Z}[\pi_1])$$

is the zero element of the group.

Proof. If X is a finite complex, then we take $K = X$ and φ to be the identity map, which would clearly give us the zero element in $\tilde{K}_0(\mathbb{Z}[\pi_1])$.
Now, suppose K is a finite n-complex dominating X and we constructed an $n-1$ connected map $\varphi : K \to X$ such that $[\pi_n(\varphi)] = 0 \in \tilde{K}_0(\mathbb{Z}[\pi_1])$. Then, there exists a finitely generated free $\mathbb{Z}[\pi_1]$-module F such that $[\pi_n(\varphi)] \oplus F$ is free. Construct a bouquet of $n-1$ spheres, attach them to K and extend φ by mapping them to the base point of $\pi_1(X)$. Now, $\pi_n(\varphi)$ is free by arguments made in previous proofs, and so we get a homotopy equivalence by the lemma in the previous subsection. ☐

4. A More Algebraic Understanding

Instead of defining the finiteness obstruction for a CW complex, we now define the obstruction for a *chain complex*. This will allow us to apply obstruction theory in larger contexts.

Definition 4.1. Given a chain complex C_*, we define $C^{(n)}$ to be the n-skeleton, i.e. the chain complex $0 \to C_n \to \cdots \to C_1 \to 0$. Further, C_* is *bounded above* if there exists $k \in \mathbb{N}$ such that $C_* = C^{(k)}$

Now, we define a domination of chain complexes:

Definition 4.2. Given A_* and C_* that are finitely generated free R chain complexes for some ring R, we define C_* to *dominated* by A_* if there exists a sequence of module homomorphisms $h_i : A_i \to C_i$ such that each h_i has a right inverse chain map k_i. Further, we say that C_* is finitely dominated if A_* is bounded above.

We see how this form of domination coincides with the domination in the previous section.

Proposition 4.3. *Suppose X satisfies Dn and is finitely dominated by an n complex K, and we construct an n-connected map $\varphi : K \to X$. We observe that φ further induces a finite domination of $C_i(\tilde{X})$ by $C_i(\tilde{K})$ by inducing $\mathbb{Z}[\pi]$-homomorphisms $C_i(\tilde{K}) \to C_i(\tilde{X})$ with right inverses.*

It should be clear that φ induces such homomorphisms by the Hurewicz theorem. The right inverse homomorphisms are induced by the right inverses we constructed in Lemma 3.8. Further, since X satisfies Dn, all of the higher homology groups of $\tilde{X}$ are zero. We can now make sense of the wall obstruction for an arbitrary chain complex.

Definition 4.4. Given an R-free chain complex C_* such that the C_i are finitely generated for each i. If the inclusion $C^{(n-1)} \hookrightarrow C$ is an R domination, then $H_n(C, C^{(n)})$ is a finitely generated projective R module. We can now define the wall invariant as:

$$w(C) = [H_n(C, C^{(n)})] \in \tilde{K}_0(R)$$

We conclude with the propositon:

Proposition 4.5. *Given a chain complex with the properties mentioned in the previous definition, $w(C) = 0$ iff C is R-chain equivalent to a finite R-free complex.*

Observe that in the case of a CW-complex, $R = \mathbb{Z}[\pi_1]$, $C = C_*(\tilde{X})$, and our $n-1$ connected map φ induces the finite domination of chain complexes. Further, $H_n(C, C^{(n-1)}) = \pi_n(\varphi)$ by the Hurewicz theorem. Now, we provided a framework for discussing obstruction in any homology theory.

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